

Fuzzy semiprime subsets of ordered groupoids (groupoids)

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Abstract

A fuzzy subset f of an ordered semigroup (or semigroup) S is called fuzzy semiprime if $f(x) \geq f(x^2)$ for every $x \in S$ (Definition 1). Following the terminology of semiprime subsets of ordered semigroups (semigroups), the terminology of ideal elements of *poe*-semigroups (: ordered semigroups possessing a greatest element), and the terminology of ordered semigroups, in general, a fuzzy subset f of an ordered semigroups (semigroup) should be called fuzzy semiprime if for every fuzzy subset g of S such that $g^2 := g \circ g \preceq f$, we have $g \preceq f$ (Definition 2). And this is because if S is a semigroup or ordered semigroup, then the set of all fuzzy subsets of S is a semigroup (ordered semigroup) as well. What is the relation between these two definitions? that is between the usual definition (Definition 1) we always use and the definition we give in the present paper (Definition 2) saying that that definition should actually be the correct one? The present paper gives the related answer.

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1 Introduction

An ordered groupoid (: *po*-groupoid), denoted by (S, \cdot, \leq) , is an ordered set (S, \leq) endowed with a multiplication " \cdot " which is compatible with the ordering (that is, $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for every $c \in S$). If this multiplication is associative, then S is called an ordered semigroup (: *po*-semigroup) [1, 2]. A *poe*-semigroup is an ordered semigroup having a greatest element usually denoted by " e " ($e \geq a$ for all $a \in S$) (cf. for example [3]). Following L. Zadeh [4], the founder of fuzzy sets, if S is an ordered groupoid (or groupoid), a fuzzy subset of S (or a fuzzy set in S) is a mapping f of S into the closed interval $[0, 1]$ of real numbers. For a nonempty subset A of an ordered groupoid (or groupoid) S , the characteristic function f_A is the fuzzy subset on S defined by

$$f_A : S \rightarrow \{0, 1\} \mid x \rightarrow \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

If (S, \cdot) is a groupoid, f, g fuzzy subsets of S and $x \in S$, we define

$$(f \circ g)(x) := \begin{cases} \sup_{y, z \in S, yz=x} [\min\{f(y), g(z)\}] & \text{if there exist } y, z \in S \text{ such that } x = yz \\ 0 & \text{if there are no } y, z \in S \text{ such that } x = yz. \end{cases}$$

If S is an ordered groupoid (or groupoid), $x \in S$ and $\lambda \in [0, 1]$, the mapping

$$x_\lambda : S \rightarrow [0, 1] \mid y \rightarrow \begin{cases} \lambda & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is called a *fuzzy point* of S . For short, we write x_λ^2 instead of $(x^2)_\lambda$.

If S is an ordered groupoid, then for an element a of S , we define

$$A_a = \{(x, y) \in S \times S \mid a \leq xy\}.$$

For two fuzzy subsets f, g of S , we define the multiplication $f \circ g$ as:

$$(f \circ g)(a) := \begin{cases} \bigvee_{(x, y) \in A_a} \min\{f(x), g(y)\} & \text{if } A_a \neq \emptyset \\ 0 & \text{if } A_a = \emptyset, \end{cases}$$

and the order relation as follows:

$$f \preceq g \text{ if and only if } f(x) \leq g(x) \text{ for all } x \in S.$$

If f, g are fuzzy subsets of S such that $f \preceq g$ then, for every fuzzy subset h of S , we have $f \circ h \preceq g \circ h$ and $h \circ f \preceq h \circ g$. If S is an ordered semigroup, then the multiplication of fuzzy subsets of S is associative, so the set of all fuzzy subsets of S with the multiplication and the order above is an ordered semigroup, in particular, a *poe*-semigroup [5]. If S is an ordered groupoid, then the set of all fuzzy subsets of S is a *poe*-groupoid [5]. Just for an information, if S is an ordered groupoid (resp. ordered semigroup), then the *poe*-groupoid (resp. *poe*-semigroup) of all fuzzy subsets of S has a zero element and S is embedded in the set of all fuzzy subsets of S [5].

According to Clifford and Preston [6; p. 121], a subset T of a semigroup S is called semiprime if for every $a \in S$ such that $a^2 \in T$, we have $a \in T$. Semiprime ideals play an important role in studying the structure of semigroups. As an example, a semigroup S is left regular (resp. right regular) if and only if every left (resp. right) ideal of S is semiprime. A semigroup S is intra-regular if and only if every ideal (that is, two-sided ideal) of S is semiprime. These, in turn, are equivalent to saying that a semigroup S is left regular if and only if it is a

union (or disjoint union) of left simple subsemigroups of S (the right analogue also holds). Every left and every right ideal of S is semiprime if and only if S is union of groups (or disjoint groups) (which means that every left and every right ideal of S is semiprime). A semigroup S is intra-regular (which means that the principal ideals of S constitute a semilattice Y under intersection) if and only if it is a union of simple semigroups. The semiprime subsets of ordered semigroups (groupoids) have been defined in [7] in the same way. According to [7], a subset T of an ordered semigroup S is called semiprime if for every $A \subseteq T$ such that $A^2 \subseteq T$, we have $A \subseteq T$ (which actually is the same with that one given by Clifford and Preston as the two definitions are equivalent). In a series of papers the authors of the present paper have shown that, exactly as in semigroups, semiprime subsets of ordered semigroups play an important role in studying the structure of ordered semigroups.

A fuzzy subset f of a semigroup S is called *semiprime* if $f(x) \geq f(x^2)$ for every $x \in S$. This concept has been first introduced by N. Kuroki in [8], as he was the first who observed and showed in [8] that a nonempty subset A of a semigroup S is semiprime if and only if its characteristic function f_A is fuzzy semiprime. Kehayopulu and Tsingelis were the first who studied fuzzy ordered groupoids [9]. Following Kuroki, they kept the same definition of semiprime subset of an ordered groupoid as a fuzzy subset f of S satisfying $f(x) \geq f(x^2)$ for every $x \in S$ [9]. Many papers on semigroups and ordered semigroups appeared adapting this definition as the definition of semiprime fuzzy subsets both for semigroups and ordered semigroups. It might be also noted that a fuzzy subset f of a groupoid S is semiprime if and only if for every $x \in S$ and every $\lambda \in [0, 1]$ such that $x_\lambda \circ x_\lambda \leq f$ implies $x_\lambda \leq f$ [10].

On the other hand, an element t of a *poe*-groupoid (or *poe*-semigroup) S is called semiprime if for every $a \in S$ such that $a^2 \leq t$, we have $a \leq t$ [3]. And the same definition of semiprime elements is the usual definition for ordered semigroups in general. As this is the case for ordered semigroups, in addition, since the fuzzy subsets of ordered semigroups form an ordered semigroup, one should expect that in the theory of fuzzy ordered semigroups (or semigroups) the fuzzy semiprime subset should be defined in a similar way. That is, if S is an ordered semigroup (or semigroup), then a fuzzy subset f of S should be called fuzzy semiprime if for any fuzzy subset g of S such that $g \circ g \preceq f$, we have $g \preceq f$. However in the existing bibliography, for an ordered semigroup S , a fuzzy subset f of S is called fuzzy semiprime if $f(x) \geq f(x^2)$ for every $x \in S$ (cf. for example [8–12]) and this is the usual definition the authors always use. It is natural to ask what is the relation between these two definitions. The present paper gives the related answer. Here we prove that if a fuzzy subset f of an ordered groupoid (semigroup) S is semiprime (in the usual sense), then for any fuzzy subset g of S such that $g \circ g \preceq f$, we have $g \preceq f$, and that the converse statement does not hold in general. However, for the ordered

semigroups satisfying the condition

$$x \leq yz \implies \min\{f(y^2), f(z^2)\} \leq f(x),$$

the two definitions are equivalent.

2 Main results

Proposition 1. *Let (S, \cdot, \leq) be an ordered groupoid and f a fuzzy subset of S . Then*

$$f(x) \leq (f \circ f)(x^2) \text{ for every } x \in S.$$

Proof. Let $x \in S$. Since $(x, x) \in A_{x^2}$, we have $A_{x^2} \neq \emptyset$ and

$$\begin{aligned} (f \circ f)(x^2) &= \bigvee_{(u,v) \in A_{x^2}} \min\{f(u), g(v)\} \\ &\geq \min\{f(x), f(x)\} \\ &= f(x), \end{aligned}$$

so $f(x) \leq (f \circ f)(x^2)$. □

Proposition 2. *Let (S, \cdot, \leq) be an ordered groupoid and f, g fuzzy subsets of S such that $g \circ g \preceq f$. Then*

$$g(x) \leq f(x^2) \text{ for every } x \in S.$$

Proof. Let $x \in S$. Since g is a fuzzy subset of S , by Proposition 1, we have $g(x) \leq (g \circ g)(x^2)$. Since $g \circ g \preceq f$, we have $(g \circ g)(x^2) \leq f(x^2)$. Thus we have $g(x) \leq f(x^2)$. □

Definition 3. [9] If S is an ordered groupoid, a fuzzy subset f of S is called *fuzzy semiprime* if $f(x) \geq f(x^2)$ for every $x \in S$.

Theorem 4. *Let S be an ordered groupoid and f a fuzzy subset of S . We consider the following statements:*

(1) *f is fuzzy semiprime.*

(2) *If g is a fuzzy subset of S such that $g \circ g \preceq f$, then $g \preceq f$.*

Then (1) \implies (2). The implication (2) \implies (1) does not hold in general.

Proof. (1) \implies (2). Let g be a fuzzy subset of S such that $g \circ g \preceq f$ and $x \in S$. By Proposition 2, we have $g(x) \leq f(x^2)$. Since f is fuzzy semiprime, we have $f(x) \geq f(x^2)$. Then we have $g(x) \leq f(x)$ and (2) holds.

Condition (2) does not always imply (1). In fact:

The set $S = \{n \in \mathbb{N} \mid n \geq 2\} = \{2, 3, 4, \dots\}$ of natural numbers with the usual multiplication and the usual order is an ordered groupoid (in particular, it is an ordered semigroup). Let f be the fuzzy subset of S defined by:

$$f : (S, \cdot, \leq) \rightarrow [0, 1] \mid x \rightarrow \begin{cases} 0 & \text{if } x = 2 \\ 1 & \text{if } x > 2. \end{cases}$$

Condition (2) is satisfied. Indeed: Let g be a fuzzy subset of S such that $g \circ g \preceq f$ and let $x \in S$.

(I) Let $x = 2$.

We consider the set $A_2 = \{(m, n) \in S \times S \mid 2 \leq mn\}$. Since $(2, 2) \in A_2$, we have $A_2 \neq \emptyset$ and

$$\begin{aligned} (g \circ g)(2) &= \bigvee_{(m,n) \in A_2} \min\{g(m), g(n)\} \\ &\geq \min\{g(2), g(2)\} \\ &= g(2), \end{aligned}$$

Since $g \circ g \preceq f$, we have $(g \circ g)(2) \leq f(2)$, so we have $g(2) \leq f(2)$.

(II) Let $x > 2$. Then $f(x) = 1$. On the other hand, since g is a fuzzy subset of S , we have $g(x) \leq 1$. Thus we have $g(x) \leq f(x)$.

By (I) and (II), we have $g(x) \leq f(x)$. Since this holds for any $x \in S$, we have $g \preceq f$.

Condition (1) does not hold. In fact, we have $f(2) = 0$ and $f(2^2) = f(4) = 1$, so $f(2) \not\geq f(2^2)$. \square

It is natural to ask under what conditions the implication $(2) \Rightarrow (1)$ is satisfied. The next theorem gives a related answer.

Theorem 5. *Let S be an ordered groupoid and f a fuzzy subset of S such that*

- (a) $a \leq xy \implies \min\{f(x^2), f(y^2)\} \leq f(a)$ ($a, x, y \in S$) and
- (b) *if g is a fuzzy subset of S such that $g \circ g \preceq f$, then $g \preceq f$.*

Then f is fuzzy semiprime.

Proof. Let $x \in S$. We consider the fuzzy subset g of S defined by:

$$g : (S, \cdot, \leq) \rightarrow [0, 1] \mid x \rightarrow f(x^2).$$

We have $g \circ g \preceq f$. In fact: Let $a \in S$. If $A_a = \emptyset$, then $(g \circ g)(a) = 0 \leq f(a)$. If $A_a \neq \emptyset$, then

$$(g \circ g)(a) = \bigvee_{(x,y) \in A_a} \min\{g(x), g(y)\}.$$

We have

$$\min\{g(x), g(y)\} \leq f(a) \text{ for every } (x, y) \in A_a.$$

Indeed, if $(x, y) \in A_a$, then $a \leq xy$ and, by (a), $\min\{f(x^2), f(y^2)\} \leq f(a)$, that is $\min\{g(x), g(y)\} \leq f(a)$. Therefore we have $(g \circ g)(a) \leq f(a)$. This is for every $a \in S$, thus we obtain $g \circ g \preceq f$. By condition (a), we get $g \preceq f$, then $f(x) \geq g(x) = f(x^2)$. This holds for every $x \in S$, so S is fuzzy semiprime. \square

Remark 6. Fuzzy semiprime subsets of ordered semigroups do not satisfy the condition (a) of Theorem 5 in general. In fact: Let $S = [0, 1]$ be the ordered semigroup of real numbers with the usual multiplication and the usual order of reals and f the fuzzy subset on S defined by

$$f : (S, \cdot, \leq) \rightarrow [0, 1] \mid x \rightarrow x$$

(that is, the identity mapping on S). Then f is fuzzy semiprime. Indeed: If $x \in S$, then $x < 1$. Since $x \geq 0$, we have $x^2 \leq x$, so $f(x) \geq f(x^2)$. S does not satisfy the condition (a). In fact:

$$\frac{1}{10} \leq \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3}$$

but

$$\begin{aligned} \min\{f((\frac{1}{2})^2), f((\frac{1}{3})^2)\} &= \min\{f(\frac{1}{4}), f(\frac{1}{9})\} = \min\{\frac{1}{4}, \frac{1}{9}\} \\ &= \frac{1}{9} \not\geq \frac{1}{10} = f(\frac{1}{10}). \end{aligned}$$

3 Conclusion

As a conclusion, let us give the two definitions below: The first one is the definition in the existing bibliography we always use. The second is similar with the definition of semiprime subsets (or ideal elements) of ordered groupoid. The definition which should actually be.

In the following, S is an ordered groupoid (or groupoid) and $g^2 := g \circ g$.

Definition 1. A fuzzy subset f of S is called *fuzzy semiprime* if

$$f(x) \geq f(x^2)$$

for every $x \in S$.

Definition 2. A fuzzy subset f of S is called *fuzzy semiprime* if

For any fuzzy subset g of S such that $g^2 \preceq f$, we have $g \preceq f$.

Then Definition 1 implies Definition 2, but Definition 2 does not imply Definition 1 in general. In particular, if the fuzzy subset f of S has the property

$$a \leq xy \implies \min\{f(x^2), f(y^2)\} \leq f(a),$$

then the two definitions are equivalent.

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